On the necessity of new ramification breaks

Nigel P. Byott [a], G. Griffith Elder [b,*,1]

[a] School of Engineering, Computer Science and Mathematics, University of Exeter, Exeter EX4 4QE U.K.

[b] Department of Mathematics, Virginia Tech, Blacksburg VA 24061-0123 U.S.A.

Abstract

Ramification invariants are necessary, but not in general sufficient, to determine the Galois module structure of ideals in local number field extensions. This insufficiency is associated with elementary abelian extensions, where one can define a refined ramification filtration – one with more ramification breaks (BE05). The first refined break number comes from the usual ramification filtration and is therefore necessary. Here we study the second refined break number.

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1 Introduction

Let p be a prime integer, and let K be a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, with absolute ramification index e_K and inertia degree f. Let N be a finite, fully ramified, Galois p-extension of K, let $G = \mathrm{Gal}(N/K)$, and let $\mathfrak{P}_N$ be the maximal ideal of the valuation ring $\mathfrak{O}_N$ of N. Also, let T be the maximal unramified subfield of K. Thus the valuation ring $\mathfrak{O}_T$ of T is the ring of Witt vectors of $\mathbb{F}_q$, where $q = p^f$. It is natural to ask about the structure of each ideal $\mathfrak{P}_N^r$ under the canonical action of the group ring $\mathfrak{O}_T[G]$. This question has its roots in the Normal Basis Theorem, see e.g. (Lan84, p. 344), and in the Normal Integral Basis Theorem of E. Noether (Noe32).

* Corresponding Author
Email addresses: N.P.Byott@ex.ac.uk (Nigel P. Byott), elder@vt.edu (G. Griffith Elder).

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Complexity, however, threatens to overwhelm any complete, explicit description, even when one restricts oneself to relatively simple Galois groups (Eld95; Eld02; Eld06). So instead, we ask for those invariants upon which the structure depends. Certainly these must include those associated with the usual ramification filtration

$$G_i = \{\sigma \in G : (\sigma - 1)\mathfrak{P}_N \subset \mathfrak{P}_N^{i+1}\}.$$

For example, it is easily shown that the ramification break numbers (that is, the integers b such that $G_b \supsetneq G_{b+1}$) are necessary to determine the Galois module structure of the ideals of $\mathfrak{O}_N$. To see this, simply consider the structure of the ideal fixed by G_{b+1}, namely $(\mathfrak{P}_N^r)^{G_{b+1}}$, over the group ring $\mathfrak{O}_T[\sigma]$ for some $\sigma \in G_b \setminus G_{b+1}$. Since $G_b/G_{b+1} \cong C_p^s$ is necessarily elementary abelian (Ser79, IV §2 Prop 7 Cor 3), this is a module over the cyclic group ring $\mathfrak{O}_T[C_p]$, for which there are exactly three indecomposable modules: the trivial module $\mathfrak{O}_T$, the group ring or regular representation $\mathfrak{O}_T[C_p]$, and the module $\mathfrak{O}_T[\zeta_p]$ where σ acts via multiplication by the p-th root of unity ζ_p (CR90, Thm 34.31). Now proceed as in (RCVSM90, Thm 1) to see how the multiplicities of these three modules are parametrized by b (along with the absolute ramification degree).

The usual ramification invariants are not however sufficient to determine the Galois module structure of ideals. This was observed in (BE02) where we considered biquadratic extensions (the case $p = 2$) with one break. The work presented here, together with (BE05), stems from our ongoing effort to fully understand the implications of that paper, and to extend its results to arbitrary p. With hindsight we can now say that the insufficiency of the usual ramification filtration is tied to the elementary abelian quotients of consecutive ramification groups G_b/G_{b+1}, but that there is a 'repair'. We can focus on the elementary abelian extension with Galois group G_b/G_{b+1} and define a new refined ramification filtration, one with more information – more breaks (BE05). In this paper, we amend the definition from (BE05) slightly; study the necessity, for the Galois module structure of ideals, of the first piece of new information that this refined ramification filtration provides – the second refined break; and explicitly describe the Galois module structure of ideals in bicyclic extensions under *maximal refined ramification*, when this second refined break achieves a natural upper bound.

1.1 Refined Ramification Filtration

Let N/K be a fully ramified, elementary abelian p-extension with one break in its ramification filtration, at b. So $G = \mathrm{Gal}(N/K) \cong G_b/G_{b+1}$. Note that G is a vector space over $\mathbb{F}_p$, the field with p elements. To enable the residue field $\mathbb{F}_q$ to act on G as well, let $\mathbb{Z}_{(p)}$ denote the integers localized at p, and define

truncated exponentiation by the polynomial

$$(1+X)^{[Y]} := \sum_{i=0}^{p-1} \binom{Y}{i} X^i \in \mathbb{Z}_{(p)}[X,Y],$$

a truncation of the usual binomial series. Now let $\mathcal{A} = (\sigma - 1 : \sigma \in G)$ denote the augmentation ideal of $\mathfrak{O}_T[G]$. If L is any finite extension of T that is contained in K, then $\mathfrak{O}_L\mathcal{A}$ is the augmentation ideal of $\mathfrak{O}_L[G]$. For any $\kappa \in \mathfrak{O}_L$ and any $x \in 1 + \mathfrak{O}_L\mathcal{A}$, truncated exponentiation gives a well-defined element $x^{[\kappa]}$ of $1 + \mathfrak{O}_L\mathcal{A}$. This does not make $1 + \mathfrak{O}_L\mathcal{A}$ into an $\mathfrak{O}_L$-module since, for example, we do not in general have $(x^{[\kappa]})^{[\kappa']} = x^{[\kappa\kappa']}$. To address this problem we could choose to work with the quotient group $(1 + \mathfrak{O}_L\mathcal{A})/(1 + p\mathfrak{O}_L\mathcal{A})$.

This is the approach of (BE05) in the case $L = T$, where we proposed working with the quotient group $(1+\mathcal{A})/(1+p\mathcal{A})$ over the field $\mathfrak{O}_T/p\mathfrak{O}_T = \mathbb{F}_q$. As noted there, $(1+\mathcal{A})/(1+p\mathcal{A})$ is a "near-space" over $\mathbb{F}_q$: it satisfies all the properties of a vector space over $\mathbb{F}_q$ except the distributive property, $(x_1x_2)^{[\omega]} \neq x_1^{[\omega]}x_2^{[\omega]}$. In the case of biquadratic extensions, the refined ramification filtration of this near space contains extraneous information in the form of an "extra" third refined break (BE05, §4). This is undesirable and expected more generally.

So, in this paper, we propose working with the smaller group $\mathcal{G} = (1+\mathcal{A})/(1+\mathcal{A}^p)$. Notice that because G is elementary abelian, we have $p\mathcal{A} \subset \mathcal{A}^p$. Following (BE05, Thm 2.1) and the discussion leading to (BE05, Cor 2.3), we find

$$(\omega, x) \in \mathbb{F}_q \times \mathcal{G} \longrightarrow x^{[\omega]} \in \mathcal{G}$$

is an $\mathbb{F}_q$-action that endows $\mathcal{G}$ with the structure of an $\mathbb{F}_q$-vector space. Let $G^{\mathbb{F}}$ be the span of the image of G in $\mathcal{G}$. Clearly

$$G^{\mathbb{F}} \cong \mathbb{F}_q \otimes_{\mathbb{F}_p} G.$$

Now choose any $\alpha \in N$ with $v_N(\alpha) = b$. Because of (BE, Cor 4), such elements generate normal field bases and are thus valuable for Galois module structure. Following the treatment of the usual ramification filtration (Ser79, p62), define a function i_α on $\bar{x} \in G^{\mathbb{F}}$ by the formula $i_\alpha(\bar{x}) = \sup\{v_N((x-1)\alpha) : x \in 1 + \mathcal{A},\ x \cdot (1 + \mathcal{A}^p) = \bar{x}\}$. The refined ramification filtration of $G^{\mathbb{F}}$, which cannot as yet be considered canonical as it apparently depends upon a choice of α, is defined by

$$G_j^{\mathcal{F},\alpha} = \{\bar{x} \in G^{\mathbb{F}} : i_\alpha(\bar{x}) \geq v_N(\alpha) + j\}.$$

This leads to a definition of *refined breaks*: integers j such that $G_j^{\mathcal{F},\alpha} \supsetneq G_{j+1}^{\mathcal{F},\alpha}$. Because of (BE, Cor 4) and by following (BE05, Thm 3.3), we see that there are exactly $\log_p |G|$ refined breaks.

The value of the first refined break is b (the usual ramification number) and so is clearly necessary for Galois module structure. The purpose of this paper is threefold:

(1) Show that the second refined break, which we call b_*, is canonical.
(2) Characterize those integers that appear as b_* in some extension.
(3) Discuss the relevance of b_* for Galois module structure.

Notice that we can repeat the procedure that was just described for each bicyclic subgroup $H \cong C_p^2$ of G. In each case there will be two refined breaks: b and a second refined break b_H. Since the second refined break associated with G is the minimum of these b_H, there is a bicyclic subgroup H with the refined breaks $b < b_*$. We can restrict our attention to this particular bicyclic extension and answer all three questions. Since the implications for the general Galois extension should be clear, we henceforth restrict our attention to N/K, a bicyclic extension with $G = \mathrm{Gal}(N/K) \cong C_p^2$ and refined breaks $b < b_*$.

1.2 Outline

In §2 we determine the value of b_*, find that it is canonical and moreover, that it satisfies $b < b_* \leq pb$ with the additional condition that $b_* \equiv b \bmod p$ when $b_* < pb$. The special case when $b_* = pb$ will be called *maximal refined ramification* (MRR) and $(p-1+1/p)b < b_* < pb$, *near maximal refined ramification* (NMRR). In §3 prove two results in Galois module structure. We find in Theorem 12 of §3.1 that outside of NMRR, the $\mathbb{F}_q[G]$-structure of $\mathfrak{P}_N^r/p\mathfrak{P}_N^r$ *depends* upon b_*, and therefore so too does the $\mathfrak{O}_T[G]$-structure of $\mathfrak{P}_N^r$. This addresses the question raised in the title of this paper by proving that the second refined break is necessary for the Galois module structure of ideals, as long as it is "not too big" relative to b. Then in §3.2 we show in Theorem 18 how MRR allows an easy, rather transparent and explicit description of Galois module structure in terms of $\mathfrak{O}_T[G]$-ideals.

2 The Refined Ramification Filtration in Bicyclic Extensions

Let N/K be a fully ramified bicyclic extension with $G = \mathrm{Gal}(N/K) \cong C_p^2$ and one ramification break at b, which necessarily satisfies $0 < b < pe_K/(p-1)$ and $\gcd(b,p) = 1$. We begin a process now that will define an integer, our candidate for the second refined break.

Choose $\rho_0 \in N$ with $v_N(\rho_0) = b$, and choose a pair of generators γ, σ, so that $G = \langle \gamma, \sigma \rangle$. Since $v_N((\gamma-1)\rho_0) = v_N((\sigma-1)\rho_0) = 2b$ and N/T is fully ramified,

there is a unique p^f-1 root of unity $\omega_{\gamma,\sigma}$ such that $(\gamma-1)\rho_0 \equiv \omega_{\gamma,\sigma}(\sigma-1)\rho_0 \bmod \mathfrak{P}_N^{2b+1}$. Since $\gamma \notin \langle\sigma\rangle$, $\omega_{\gamma,\sigma}^{p-1} \neq 1$. But how does $\omega_{\gamma,\sigma}$ depend upon our choice of group generators? Observe $(\gamma^i-1) = i(\gamma-1) + \sum_{j=2}^{i}\binom{i}{j}(\gamma-1)^j$ and that similarly $(\sigma^j-1) \equiv j(\sigma-1)$ modulo higher powers of $(\sigma-1)$. Moreover $(\gamma^i\sigma^j-1) = (\gamma^i-1)+(\sigma^j-1)+(\gamma^i-1)(\sigma^j-1)$. As a result, $(\gamma^i\sigma^j-1)\rho_0 \equiv i(\gamma-1)\rho_0 + j(\sigma-1)\rho_0 \equiv (i\omega_{\gamma,\sigma}+j)(\sigma-1)\rho_0 \bmod \mathfrak{P}_N^{2b+1}$. This means that the change of group generators $\langle\gamma,\sigma\rangle = \langle\gamma^a\sigma^b, \gamma^c\sigma^d\rangle$, resulting from

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{F}_p),$$

leads to $(\gamma^a\sigma^b-1)\rho_0 \equiv \omega_{\gamma^a\sigma^b,\gamma^c\sigma^d}(\gamma^c\sigma^d-1)\rho_0 \bmod \mathfrak{P}_N^{2b+1}$ and thus $a\omega_{\gamma,\sigma}+b \equiv \omega_{\gamma^a\sigma^b,\gamma^c\sigma^d}(c\omega_{\gamma,\sigma}+d) \bmod \mathfrak{P}_T$. In other words, if we identify the p^f-1 roots of unity with the nonzero elements of the finite field $\mathbb{F}_q$, we have

$$\omega_{\gamma^a\sigma^b,\gamma^c\sigma^d} = \frac{a\omega_{\gamma,\sigma}+b}{c\omega_{\gamma,\sigma}+d}.$$

A unified approach requires that we identify these roots of unity with points on the projective line, $(\omega_{\gamma^a\sigma^b,\gamma^c\sigma^d},1) = (a\omega_{\gamma,\sigma}+b, c\omega_{\gamma,\sigma}+d) \in \mathbf{P}^1(\mathbb{F}_q)$. We conclude that while the particular point $(\omega_{\gamma,\sigma},1) \in \mathbf{P}^1(\mathbb{F}_q)\setminus\mathbf{P}^1(\mathbb{F}_p)$ depends upon our choice of group generators, its orbit, $\mathrm{Orb}_{N/K} \subseteq \mathbf{P}^1(\mathbb{F}_q)\setminus\mathbf{P}^1(\mathbb{F}_p)$, under $\mathrm{PGL}_2(\mathbb{F}_p)$ is independent of both our choice of group generators and element ρ_0, and should be considered a basic invariant of the extension.

Fix $\rho_0 \in N$ now with $v_N(\rho_0) = b$, and fix our group generators, so $G = \langle\gamma,\sigma\rangle$. Rewrite the equation $(\gamma-1)\rho_0 \equiv \omega_{\gamma,\sigma}(\sigma-1)\rho_0 \bmod \mathfrak{P}_N^{2b+1}$ as $\gamma\rho_0 \equiv (1+\omega_{\gamma,\sigma}(\sigma-1))\,\rho_0 \bmod \mathfrak{P}_N^{2b+1}$. Motivated by the appearance of the first two terms in truncated exponentiation, we drop subscripts, write $\omega = -\omega_{\gamma,\sigma}$, and define $\Theta = \gamma\sigma^{[\omega]} \in \mathfrak{O}_T[G]$.

Observe that $(\Theta-1)\rho_0 \equiv 0 \bmod \mathfrak{P}_N^{2b+1}$. Define our "candidate" second refined break by

$$b_* := v_N((\Theta-1)\rho_0) - v_N(\rho_0).$$

This is an integer $> b$, which may depend upon our choices: of group generators and of ρ_0. Let $L = N^\sigma$ be the fixed field of $\langle\sigma\rangle$.

The purpose of this paper, as stated in §1.1, is to address three goals. In §2.1, we address the first goal by proving that b_* is the second refined break and that it is also canonical (independent of our choice of ρ_0 and also of our choice of the generators for G). In §2.2, we address the second goal by determining all realizable second refined breaks. The third goal is addressed in §3.

We begin by establishing the upper bound $b_* \leq pb$. Recall the augmentation ideal $\mathcal{A} = (\sigma - 1, \gamma - 1) \subseteq \mathfrak{O}_T[G]$ as defined in §1.1.

Lemma 1 *Given $a \not\equiv -1 \bmod p$, $\rho \in N$ with $v_N(\rho) = (1 + ap)b$, $\kappa \in \mathfrak{O}_L$ and $\mu \in \mathcal{A}^p$. Then $b \leq v_N((\gamma\sigma^{[\kappa]}(1 + \mu) - 1)\rho) - v_N(\rho) \leq pb$.*

PROOF. We need to prove two inequalities. The first is obvious. So consider the second inequality and the effect of the trace $\mathrm{Tr}_{N/L} = \Phi_p(\sigma)$ on ρ and on $\rho_* = (\gamma\sigma^{[\kappa]}(1+\mu)-1)\rho$. Because of (Ser79, V§3 Lem 4), if $v_N(\rho_*) > v_N(\rho)+pb$, then $v_L(\mathrm{Tr}_{N/L}\rho_*) > v_L(\mathrm{Tr}_{N/L}\rho) + b$.

So we prove $v_L(\mathrm{Tr}_{N/L}\rho_*) = v_L(\mathrm{Tr}_{N/L}\rho) + b$. Since $v_N((\sigma - 1)\alpha) = v_N(\alpha) + b$ if $\gcd(v_N(\alpha), p) = 1$, we have $v_N((\sigma - 1)^{p-1}\rho) = v_N(\rho) + (p - 1)b = (1 + a)pb \not\equiv 0 \bmod p^2$. Since the cyclotomic polynomial $\Phi_p(\sigma) \equiv (\sigma - 1)^{p-1} \bmod p$ and $(p - 1)b < v_N(p)$, we therefore also have $v_N(\mathrm{Tr}_{N/L}\rho) = (1 + a)pb$. So $\gcd(v_L(\mathrm{Tr}_{N/L}\rho), p) = 1$ and thus $v_L((\gamma - 1)\mathrm{Tr}_{N/L}\rho) = v_L(\mathrm{Tr}_{N/L}\rho) + b$. Notice that $\mathrm{Tr}_{N/L}\rho_* \equiv (\gamma-1)\mathrm{Tr}_{N/L}\rho \bmod p(\gamma-1)\mathrm{Tr}_{N/L}\rho$. Thus $v_L(\mathrm{Tr}_{N/L}\rho_*) = v_L(\mathrm{Tr}_{N/L}\rho) + b$ as well. □

We next establish that b_* is independent of our choice of group generators: that a change from $\langle\gamma, \sigma\rangle$ to $\langle\gamma^a\sigma^b, \gamma^c\sigma^d\rangle$ does not effect b_*, and so we have the identity $v_N((\Theta' - 1)\rho_0) - v_N(\rho_0) = b_*$ where $\Theta' = (\gamma^a\sigma^b)(\gamma^c\sigma^d)^{[-(\omega_{\gamma^a\sigma^b,\gamma^c\sigma^d})]}$. Let $\omega' = (c\omega_{\gamma,\sigma}+d)/(ad-bc)$. Using the fact that $G^{\mathbb{F}}$ is a vector space over $\mathbb{F}_q$, we have $(\Theta')^{[\omega']} = ((\gamma^a\sigma^b)^{[c\omega_{\gamma,\sigma}+d]}(\gamma^c\sigma^d)^{[-(a\omega_{\gamma,\sigma}+b)]})^{[1/(ad-bc)]} = \Theta$ in $G^{\mathbb{F}}$. And so $(\Theta')^{[\omega']} \in \Theta(1 + \mathcal{A}^p)$. The following lemma allows us to ignore terms in $\mathcal{A}^p$, so $v_N(((\Theta')^{[\omega']} - 1)\rho_0) - v_N(\rho_0) = b_*$. The desired identity then follows since $((\Theta')^{[\omega']} - 1)\rho_0 \equiv \omega'(\Theta' - 1)\rho_0 \bmod (\Theta' - 1)\rho_0\mathfrak{P}_N$.

Lemma 2 *Given $\mu \in \mathcal{A}^p$ or $\mu \in (\sigma - 1)^p \subseteq \mathfrak{O}_L[\sigma]$, then for all $\rho \in N$,*

$$v_N(\mu\rho) > v_N(\rho) + pb.$$

In particular, when $v_N(\rho) \equiv b \bmod p$ and $\kappa_i \in \mathfrak{O}_L$, we have

$$v_N((\sigma^{[\kappa_1+\kappa_2]} - \sigma^{[\kappa_1]}\sigma^{[\kappa_2]})\rho) > v_N(\rho) + pb,$$

and so if $v_N((\gamma\sigma^{[\kappa_1+\kappa_2]}-1)\rho) = v_N(\rho)+pb$, and $\min\{v_N((\gamma\sigma^{[\kappa_1]}-1)\rho), v_N(\kappa_2(\sigma-1)\rho)\} < v_N(\rho) + pb$, then $v_N((\gamma\sigma^{[\kappa_1]} - 1)\rho) = v_N(\kappa_2(\sigma - 1)\rho)$.

PROOF. Since $v_N((\sigma - 1)\rho) \geq v_N(\rho) + b$ and $v_N((\gamma - 1)\rho) \geq v_N(\rho) + b$ with strict inequality when $v_N(\rho) \equiv 0 \bmod p$, we have $v_N(\mu\rho) > v_N(\rho) + pb$ for all

$\rho \in N$. To prove the rest of the lemma, we need $\sigma^{[\kappa_1]} \cdot \sigma^{[\kappa_2]} \equiv \sigma^{[\kappa_1+\kappa_2]} \bmod (\sigma - 1)^p$ in $\mathfrak{O}_L[G]$. So observe that $(1+X)^Y \cdot (1+X)^Z = (1+X)^{Y+Z}$ in the polynomial ring $\mathbb{Q}[X,Y,Z]/(X^p)$. Therefore $(1+X)^{[Y]} \cdot (1+X)^{[Z]} = (1+X)^{[Y+Z]}$ in $\mathbb{Z}_{(p)}[X,Y,Z]/(X^p)$. Now set $X = \sigma - 1$ to obtain the second statement. As a result, if $v_N((\gamma\sigma^{[\kappa_1+\kappa_2]} - 1)\rho) - v_N(\rho) = pb$, we have $(\gamma\sigma^{[\kappa_1]}\sigma^{[\kappa_2]} - 1)\rho \equiv 0 \bmod \rho\mathfrak{P}_N^{pb}$ and $(\gamma\sigma^{[\kappa_1]} - 1)\rho \equiv -\gamma\sigma^{[\kappa_1]} \cdot (\sigma^{[\kappa_2]} - 1)\rho \bmod \rho\mathfrak{P}_N^{pb}$. Since $\gamma\sigma^{[\kappa_1]}$ is a unit and $(\sigma^{[\kappa_2]} - 1) = \kappa_2(\sigma - 1) + \sum_{i=2}^{p-1} \binom{\kappa_2}{i}(\sigma-1)^i$, the last statement follows. $\square$

Our final technical lemma establishes that the value of b_* is independent of our choice of ρ_0.

Lemma 3 *Given $\rho \in N$ with $v_N(\rho) \equiv b \bmod p^2$ and $\kappa \in \mathfrak{O}_L$, let $\mathcal{B} := v_N((\gamma\sigma^{[\kappa]} - 1)\rho) - v_N(\rho)$. Then for all $\rho' \in N$, and $\mu \in \mathcal{A}^p$*

$$v_N((\gamma\sigma^{[\kappa]}(1+\mu) - 1)\rho') - v_N(\rho') \geq \mathcal{B}.$$

Moreover, we have equality in the following cases:

(i) $\mathcal{B} = pb$, $v_N(\rho') \equiv b \bmod p$, *but* $v_N(\rho') \not\equiv (1-p)b \bmod p^2$,
(ii) $\mathcal{B} < pb$ *and* $v_N(\rho') \equiv b \bmod p^2$.
(iii) $\mathcal{B} \equiv b \bmod p$ *and* $v_N(\rho') \not\equiv 0 \bmod p$.

PROOF. Write $(\gamma\sigma^{[\kappa]}(1+\mu) - 1)\rho' = A + B$ where $A = \gamma\sigma^{[\kappa]}\mu\rho'$ and $B = (\gamma\sigma^{[\kappa]} - 1)\rho'$. By Lemma 2, $v_N(A) > v_N(\rho') + pb$. And so by Lemma 1, $v_N(A) > v_N(\rho') + \mathcal{B}$. We are left to prove $v_N(B) \geq v_N(\rho') + \mathcal{B}$, with equality in cases (i)–(iii).

We express ρ' in terms of ρ. Notice that since $\{v_N((\sigma-1)^i\rho) : i = 0, \ldots, p-1\}$ is a complete set of residues modulo p and N/L is fully ramified, there are $a_i \in L$ such that $\rho' = \sum_{i=0}^{p-1} a_i(\sigma-1)^i\rho$. Choose i_0 such that $v_N(\rho') = v_N(a_{i_0}) + i_0 b + v_N(\rho) \equiv (i_0+1)b \bmod p$. So $v_N(a_{i_0}(\sigma-1)^{i_0}\rho) = v_N(\rho')$. Note that for $i \neq i_0$, $v_N(a_i(\sigma-1)^i\rho) > v_N(\rho')$ and so $v_N(a_i) + ib > v_N(a_{i_0}) + i_0 b$. For each i, let $(\gamma\sigma^{[\kappa]} - 1) \cdot a_i(\sigma-1)^i\rho = A_i + B_i$ where $A_i = [(\gamma-1)a_i] \cdot \gamma\sigma^{[\kappa]}(\sigma-1)^i\rho$ and $B_i = a_i(\sigma-1)^i \cdot (\gamma\sigma^{[\kappa]} - 1)\rho$. This means that $B = \sum_{i=0}^{p-1}(A_i + B_i)$. Our goal is to prove that $v_N(\sum_{i=0}^{p-1}(A_i + B_i)) \geq v_N(\rho') + \mathcal{B}$.

Begin with the A_i. Notice that since $\gamma\sigma^{[\kappa]}$ is a unit, $v_N(A_i) = v_N((\gamma-1)a_i) + v_N((\sigma-1)^i\rho)$, where $v_N((\gamma-1)a_i) \geq v_N(a_i) + pb$. So for $i \neq i_0$, we have strict inequality, $v_N(A_i) > v_N(\rho') + pb \geq v_N(\rho') + \mathcal{B}$. For $i = i_0$, we have $v_N(A_{i_0}) \geq v_N(\rho') + pb \geq v_N(\rho') + \mathcal{B}$ with strict inequality when $\mathcal{B} < pb$.

Consider the B_i. Note that since $v_N(\rho') = v_N(a_{i_0}) + i_0 b + v_N(\rho)$, we have $v_N(B_{i_0}) \geq v_N(a_{i_0}) + i_0 b + v_N((\gamma\sigma^{[\kappa]} - 1)\rho) = v_N(\rho') + \mathcal{B}$. For $i \neq i_0$ we have

$v_N(a_i)+ib > v_N(a_{i_0})+i_0b$, and so we have strict inequality $v_N(B_i) > v_N(\rho')+\mathcal{B}$.

When do we have equality in the statement of our lemma? Case (i) is clear and follows immediately from Lemma 1. In cases (ii) and (iii) we have $\mathcal{B} < pb$, and so equality occurs precisely when $v_N(B_{i_0}) = v_N(\rho') + \mathcal{B}$, which occurs if and only if $v_N((\sigma-1)^j(\gamma\sigma^{[\kappa]}-1)\rho) \not\equiv 0 \bmod p$ for each $0 \leq j \leq i_0-1$. There are two extreme cases where this condition is easy to check. when $i_0 = 0$, the condition is empty. This is case (ii). When $\mathcal{B} \equiv b \bmod p$, we have $v_N((\gamma\sigma^{[\kappa]} - 1)\rho) \equiv 2b \bmod p$ and so $v_N((\sigma-1)^j(\gamma\sigma^{[\kappa]} - 1)\rho) \not\equiv 0 \bmod p$ for $0 \leq j \leq p-3$. The condition holds then if $i_0 \leq p-2$, which is equivalent to $v_N(\rho') \not\equiv 0 \bmod p$. This is case (iii). $\square$

Based upon these technical results, the integer b_* satisfies $b < b_* \leq pb$ and is canonical (independent of our choice of group generators and element $\rho \in N$ with $v_N(\rho) \equiv b \bmod p^2$). This is collected in the following theorem where we prove that it is also the second refined break, as defined in §1.1.

Theorem 4 *Let K be a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers with absolute ramification index e_K and inertia degree f. Let N/K be a fully ramified, bicyclic extension with one ramification break at b, and let $G = Gal(N/K) = \langle\gamma,\sigma\rangle$. Pick any $\rho \in N$ with $v_N(\rho) = b$. Define ω to be the unique p^f-1 root of unity such that $v_N((\gamma-1)\rho + \omega(\sigma-1)\rho) > 2b$, and let $\Theta = \gamma\sigma^{[\omega]} \in \mathfrak{O}_T[G]$. Then the refined ramification filtration has two breaks b and $b_* = v_N((\Theta-1)\rho) - v_N(\rho)$, so that*

$$G^{\mathbb{F}} = \langle\Theta,\sigma\rangle = G^{\mathbb{F},\rho}_b \supsetneq G^{\mathbb{F},\rho}_{b+1} = \langle\Theta\rangle = G^{\mathbb{F},\rho}_{b_*} \supsetneq G^{\mathbb{F},\rho}_{b_*+1} = \{e\}.$$

Moreover b_* satisfies $b < b_* \leq pb$ and is independent of our choices..

PROOF. By Lemma 1, $b < b_* \leq pb$. Let $\bar{\Theta}$ denote the image of Θ in $\mathcal{G}$ as defined in §1.1. By Lemma 3 cases (i) and (ii) (with $\rho' = \rho$), we have $i_\rho(\bar{\Theta}) = b + b_*$. $\square$

2.2 The value of the second refined break

The determination of all possible values of b_* will require a detour through (and detailed analysis of) Kummer bicyclic extensions with one break at b. We therefore begin by summarizing the results of this detour in the following theorem, which is a consequence of Proposition 10 and Corollary 11. Its proof appears in §2.2.4.

Theorem 5 *Let $U := pb - \max\{(p^2-1)b - p^2e_K, 0\}$. Assuming the conditions of Theorem 4, $b < b_* \leq U$, and if $b_* < U$ then $b_* \equiv b \bmod p$ but $b_* \not\equiv$*

$(1+p)b \bmod p^2$. *Moreover, any integer that satisfies these conditions is the second refined break of a bicyclic extension with one break at b.*

Corollary 6

$$pb_* - b < p^2 e_K$$

PROOF. From Theorem 5, $b_* \leq pb - \max\{(p^2-1)b - p^2 e_K, 0\}$, which leads to two cases depending upon whether or not $\max\{(p^2-1)b - p^2 e_K, 0\} = 0$. Suppose $\max\{(p^2-1)b-p^2e_K, 0\} = 0$. Thus $(p^2-1)b < p^2 e_K$ (recall $\gcd(p,b) = 1$) and $b_* \leq pb$. Then $pb_* - b \leq (p^2-1)b < p^2 e_K$. Suppose $\max\{(p^2-1)b - p^2 e_K, 0\} = (p^2-1)b - p^2 e_K$. Thus $p^2 e_K < (p^2-1)b$ and $b_* \leq p^2 e_K - (p^2-p-1)b$. So $pb_* - b \leq p^3 e_K - (p^2-1)(p-1)b < p^3 e_K - (p-1)p^2 e_K = p^2 e_K$. □

2.2.1 A brief history

The chronology of this research may be of interest. We began our investigations by looking at Kummer extensions, as we tried to generalize the results of (BE02) from $p = 2$ to $p > 2$. In the course of these investigations, truncated exponentiation appeared first within the group ring $\mathfrak{O}_T[G]$, as we worked to prove Lemma 9. It is this appearance of truncated exponentiation that led us to the investigations in (BE05), and to Lemmas 1, 2 and 3 and Theorem 4. Only later as we worked to determine the precise value of b_*, did truncated exponentiation emerge among the generators of the bicyclic Kummer extension. This work is captured in Proposition 10 below. Our presentation here reverses that chronology somewhat, as we start in §2.2.2 by assuming truncated exponentiation among the generators of our extension.

2.2.2 Bicyclic Kummer extensions with one break

Let ζ denote a nontrivial pth root of unity, and assume that $\zeta \in K$. Given any integer b such that $0 < b < pe_K/(p-1)$ with $\gcd(b,p) = 1$, choose $\beta \in K$ such that $v_K(\beta) = pe_K/(p-1) - b$. Choose a $p^f - 1$ root of unity ω such that $\omega^{p-1} \neq 1$, and set

$$x^p = 1 + \beta, \quad y^p = (1+\beta)^{[\omega^p]}.$$

For either $t = 0$ or $0 < t < b$ with $\gcd(t,p) = 1$, choose $\tau \in K$ such that $v_K(\tau) = pe_K/(p-1) - t$. Set

$$z^p = 1 + \tau.$$

Then $N_z := K(x, yz)$, a subfield of $K(x,y,z)$, is a fully ramified, bicyclic extension with one break in its ramification filtration, at b. Moreover, *any* fully ramified, bicyclic extension with one break can be represented in this

way. In particular, there are τ with $t=0$ such that $1+\tau$ is a pth power. In this case, we have $N_z = N_1 := K(x,y)$.

Choose $\sigma, \gamma \in G = \mathrm{Gal}(N_z/K)$ with

$$\sigma x = x, \quad \sigma yz = \zeta yz,$$
$$\gamma x = \zeta x, \; \gamma yz = yz.$$

And let $L = K(x)$.

Why have we chosen to express the generators in this way? Our first choice, to represent x^p as $1+\beta$, is natural: p-adic defects of units are related to ramification numbers (Wym69). Our second choice, to represent yz as a product, means that N_z can be seen as a 'twist' of $N_1 = K(x,y)$. See §2.2.3. Our final choice, to relate y^p to x^p by truncated exponentiation, is justified simply by the fact that it makes the nice statement in Proposition 10 possible.

We are interested in the refined ramification filtration, and so we require now an element ρ_0 of N_z with valuation b. Observe that since N_z/L is a cyclic Kummer extension with break number b, $N_z = L(Y_z)$ for some Y_z with $Y_z^p = 1+\beta_z \in L$ and $v_L(\beta_z) = p^2 e_K/(p-1) - b$. Clearly then $\rho_0 = (\zeta-1)/(Y_z-1)$ will do. Observe furthermore $L(Y_z) = L(yz)$. To describe the Galois action (and in particular the γ-action) on ρ_0 and thus on Y_z we ask that yz/Y_z be an explicitly described element in L. This is accomplished in the following two lemmas.

Lemma 7 *There is a $\beta_L \in L$ with $v_L(\beta_L) = p^2 e_K/(p-1) - b$ such that*

$$(1+\beta)^{[\omega^p]} = \left(x^{[\omega]}\right)^p \cdot (1+\beta_L).$$

PROOF. The norm, from L to K, of $x-1$ is $(-1)^{p-1}\beta$. So $v_L(x-1) = pe_K/(p-1) - b$ and thus $v_L(p(x-1)) = p^2 e_K/(p-1) - b$. Now $\left(x^{[\omega]}\right)^p \equiv \sum_{i=0}^{p-1} \binom{\omega}{i}^p (x-1)^{pi} + \omega p(x-1) \bmod p(x-1)^2$. Note that $\binom{\omega}{i}^p = \binom{\omega^p}{i}$ for $i = 0, 1$, and $\binom{\omega}{i}^p (x-1)^i \equiv \binom{\omega^p}{i}(x-1)^i \bmod p(x-1)^2$ for $i \geq 2$. Furthermore since $1+\beta = (1+(x-1))^p$, $(x-1)^p = \beta - \sum_{i=1}^{p-1}\binom{p}{i}(x-1)^i \equiv \beta - p(x-1) \bmod p(x-1)^2$. So $(x-1)^{pi} \equiv \beta^i \bmod p(x-1)^2$ for $i > 1$. Therefore $\left(x^{[\omega]}\right)^p \equiv \sum_{i=0}^{p-1}\binom{\omega^p}{i}\beta^i \cdot (1+(\omega-\omega^p)p(x-1)) \bmod p(x-1)^2$. Since $\omega \notin \mathbb{Z}_p$, $\omega - \omega^p$ is a unit. The result follows. $\square$

Lemma 8 *There are elements $\delta', \tau_L \in L$ with $v_L(\delta') = pe_K/(p-1) - t$ and $v_L(\tau_L) = p^2 e_K/(p-1) - t$ such that $1+\tau = (1+\delta')^p(1+\tau_L)$.*

PROOF. If $t = 0$ then $K(z)/K$ is unramified. Thus $L(z)/L$ is unramified and the result is clear. If $t \neq 0$ then $K(z)/K$ is ramified with ramification number t. Thus $K(x, z)$ is a fully ramified C_p^2 extension with two lower ramification numbers, $b_1 = t, b_2 = t + p(b - t)$. Since $L(z)/L$ is a Kummer, ramified C_p-extension with ramification number t, we find that $L(z) = L(Z)$ where $Z^p = 1 + \tau_L$ for some $\tau_L \in L$ with $v_L(\tau_L) = p^2 e_K/(p-1) - t$ (Wym69). Moreover, Z may be chosen so that $z/Z \in L$. In that case, $z/Z = 1 + \delta'$ for some $\delta' \in L$ with $v_L(\delta') = pe_K/(p-1) - t$. $\square$

Now using the δ' of Lemma 8, define $r_z \in L$ by

$$r_z = x^{[\omega]}(1 + \delta) \text{ where } \delta = \begin{cases} \delta' \text{ for } t > b/p, \\ 0 \text{ for } t < b/p. \end{cases} \tag{1}$$

Choose $Y_z = yz/r_z \in N_z$, so r_z is the 'ratio' $yz/Y_z \in L$ and $\sigma Y_z = \zeta Y_z$. Using Lemma 7, $Y_z^p = 1 + \beta_z$ where

$$1 + \beta_z = \begin{cases} (1 + \beta_L)(1 + \tau_L) \text{ for } t > b/p, \\ (1 + \beta_L)(1 + \tau) \text{ for } t < b/p. \end{cases}$$

As a result, $v_{N_z}(Y_z - 1) = v_L(\beta_z) = p^2 e_K/(p-1) - b$ and

$$\rho_0 = \frac{\zeta - 1}{Y_z - 1} \tag{2}$$

satisfies $v_{N_z}(\rho_0) = b$.

We now recall an earlier observation: Since $v_{N_z}((\gamma - 1)\rho_0) = v_{N_z}((\sigma - 1)\rho_0) = 2b$, there is an element $a \in \mathfrak{O}_T$ such that $(\gamma - 1)\rho_0 \equiv a(\sigma - 1)\rho_0 \bmod \mathfrak{P}_{N_z}^{2b+1}$, which can be rewritten as $\gamma\rho_0 \equiv \sigma^{[a]}\rho_0 \bmod \mathfrak{P}_{N_z}^{2b+1}$, and also as $(\gamma\sigma^{[-a]} - 1)\rho_0 \bmod \mathfrak{P}_{N_z}^{2b+1}$. We are interested in determining a along with the precise valuation, $v_{N_z}((\gamma\sigma^{[-a]} - 1)\rho_0)$. Recall the generic bounds given in Lemma 1.

Lemma 9 *Using the notation of this section,* $\gamma\sigma^{[-\Omega_z]}\rho_0 \equiv \rho_0 \bmod \rho_0^{1+p}$ *where*

$$\Omega_z := \frac{(\gamma - 1)Y_z}{(\sigma - 1)Y_z} \in \mathfrak{O}_L^*.$$

PROOF. Using the fact that $\sigma Y_z = \zeta Y_z$, we find that

$$\sigma\rho_0 = \frac{\rho_0}{1 + Y_z\rho_0} \equiv \frac{\rho_0}{1 + \rho_0} \bmod (\zeta - 1)\rho_0.$$

So we can establish by induction that

$$(\sigma-1)^t\rho_0 \equiv (-1)^t t! \prod_{i=0}^{t} \frac{\rho_0}{1+i\rho_0} \bmod \rho_0(\zeta-1) \text{ for } 0 \le t \le p-1.$$

Now define $[X]_n = X(X-1)\cdots(X-(n-1)) \in \mathbb{Z}[X]$ so that $\binom{X}{n}\cdot n! = [X]_n$ and establish the following power series identity for $\Omega \in \mathfrak{O}_L$ by induction

$$\sum_{s=0}^{t}(-1)^s[\Omega]_s \prod_{i=1}^{s}\frac{X}{1+iX} = \frac{1}{1+\Omega X}\left(1+(-1)^t[\Omega]_{t+1}\prod_{i=0}^{t}\frac{X}{1+iX}\right) \in \mathfrak{O}_L[[X]].$$

As a result,

$$\sigma^{[\Omega]}\rho_0 \equiv \frac{\rho_0}{1+\Omega\rho_0}\left(1+(\Omega^p-\Omega)\frac{\rho_0^p}{1-\rho_0^{p-1}}\right) \bmod (\zeta-1)\rho_0$$

and thus

$$\sigma^{[\Omega]}\rho_0 \equiv \frac{\rho_0}{1+\Omega\rho_0} \bmod \rho_0^{1+p}.$$

Now observe that since $v_{N_z}((\gamma-1)Y_z) = v_{N_z}(\zeta-1) = v_{N_z}((\sigma-1)Y_z))$,

$$\gamma\rho_0 = \frac{\rho_0}{1+\frac{(\gamma-1)Y_z}{\zeta-1}\rho_0} \equiv \frac{\rho_0}{1+\Omega_z\rho_0} \bmod (\zeta-1)\rho_0$$

where Ω_z is as above. Putting these together yields $\sigma^{[\Omega_z]}\rho_0 \equiv \gamma\rho_0 \bmod \rho_0^{1+p}$. By Lemma 2, $\sigma^{[-\Omega_z]}\sigma^{[\Omega_z]}\rho_0 \equiv \rho_0 \bmod \rho_0^{1+p}$. Thus the desired statement holds. □

Proposition 10 *$\Omega_z \equiv -\omega \bmod \mathfrak{P}_L$. Thus $b_* = v_{N_z}((\Theta-1)\rho_0) - v_{N_z}(\rho_0)$ where $\Theta = \gamma\sigma^{[\omega]}$. Let $\eta_z := \Omega_z + \omega \in \mathfrak{P}_L$. Then for $b_* < pb$,*

$$v_L(\eta_z) = \frac{b_*-b}{p}.$$

In general, $b_* = pb - \max\{(p^2-1)b - p^2e_K, pt-b, 0\}$.

PROOF. Recall the unit r_z. Using its definition in (1), we find that $(\gamma-1)r_z = ((\zeta x)^{[\omega]} - x^{[\omega]})(1+\delta) + (\zeta x)^{[\omega]}((\gamma-1)\delta)$. Our first observation is that since $v_L((\gamma-1)\delta) \ge pe_K/(p-1) - t + b > v_L(\zeta-1)$, we have $(\gamma-1)r_z \equiv 0 \bmod (\zeta-1)$. So using $Y_z = yz/r_z$, we can decompose Ω_z as a product: $\Omega_z = -A\cdot B$ with $A := (\gamma r_z)^{-1} \equiv r_z^{-1} \bmod (\zeta-1)$ and $B := (\gamma-1)r_z/(\zeta-1) \in \mathfrak{O}_L$.

To describe B further, we examine the term $C := (\zeta x)^{[\omega]} - x^{[\omega]}$ modulo $(\zeta-1)^2$. For $1 \le i \le p-1$, $(\zeta x-1)^i = ((\zeta-1)x+(x-1))^i \equiv i(\zeta-1)x(x-1))^{i-1} + (x-1)^i \bmod (\zeta-1)^2$. So $C \equiv (\zeta-1)\sum_{i=1}^{p-1}\binom{\omega}{i} ix(x-1)^{i-1} \bmod (\zeta-1)^2$. Observe

that $\binom{\omega}{i}i = \omega\binom{\omega-1}{i-1}$. So $C \equiv (\zeta-1)\cdot\omega x\left[x^{[\omega-1]} - \binom{\omega-1}{p-1}(x-1)^{p-1}\right] \bmod (\zeta-1)^2$. Now replace A, B and C, in the expression for Ω_z, and find

$$\Omega_z \equiv -\frac{\omega x\left(x^{[\omega-1]} - \binom{\omega-1}{p-1}(x-1)^{p-1}\right)}{x^{[\omega]}} - \frac{(\gamma-1)\delta}{\zeta-1}\frac{1}{1+\delta} \bmod (\zeta-1)$$
$$\equiv -\omega + \omega\binom{\omega-1}{p-1}(x-1)^{p-1} - \frac{(\gamma-1)\delta}{\zeta-1} \bmod \left(\zeta-1, (x-1)^p, \delta\frac{(\gamma-1)\delta}{\zeta-1}\right),$$

which proves the first assertion and establishes a congruence relation $\eta_z \equiv D - E$ with $D := \omega\binom{\omega-1}{p-1}(x-1)^{p-1}$ and $E := (\gamma-1)\delta/(\zeta-1)$. There are two cases to consider: $t < b/p$ and $t > b/p$. If $t < b/p$, $\delta = 0$ and so $E = 0$. Since $v_L(D) < v_L((x-1)^p$, we find $v_L(\eta_z) = v_L(D) = pe_K - (p-1)b$, when $v_L(\eta_z) < v_L(\zeta-1)$. On the other hand, if $t > b/p$ then we have $\gcd(t,p) = 1$ and since $v_L(D) = pe_K - (p-1)b \equiv b \bmod p$ while $v_L(E) = b - t$, $v_L(D) \not\equiv v_L(E) \bmod p$. Thus $v_L(D-E) = \min\{v_L(D), v_L(E)\}$. Since $\min\{v_L(D), v_L(E)\} < \min\{v_L(D(x-1)), v_L(\delta E)\}$, we have $v_L(\eta_z) = \min\{v_L(D), v_L(E)\} = b - \max\{p(b-e_K), t\}$, whenever $v_L(\eta_z) < v_L(\zeta-1)$.

From Lemma 1, $b_* := v_{N_z}((\gamma\sigma^{[\omega]}-1)\rho_0) - v_{N_z}(\rho_0) \le pb$. Using Lemma 2 with $\kappa_1 = \omega$ and $\kappa_2 = -\eta_z$, we find that when $b_* < pb$, we have $v_{N_z}((\gamma\sigma^{[\omega]}-1)\rho_0) = v_{N_z}(\eta_z(\sigma-1)\rho_0)$. So $b_* = pv_L(\eta_z) + b$. But then $v_L(\eta_z) < b - b/p < v_L(\zeta-1)$. So substituting our formulas for $v_L(\eta_z)$ into $b_* = pv_L(\eta_z) + b$, we find that $b_* = pb - (p^2-1)b + p^2e_K$ for $t < b/p$ and $b_* = pb - \max\{(p^2-1)b - p^2e_K), pt - b\}$ for $t > b/p$. Since $b_* \le pb$, these both agree with $b_* = pb - \max\{(p^2-1)b - p^2e_K), pt-b, 0\}$. $\square$

Corollary 11 *Let $U := pb - \max\{(p^2-1)b - p^2e_K, 0\}$. Any integer n satisfying $b < n \le U$, and if $n < U$ then $n \equiv b \bmod p$ but $n \not\equiv (1+p)b \bmod p^2$, is the second refined break for a bicyclic Kummer extension with one break at b.*

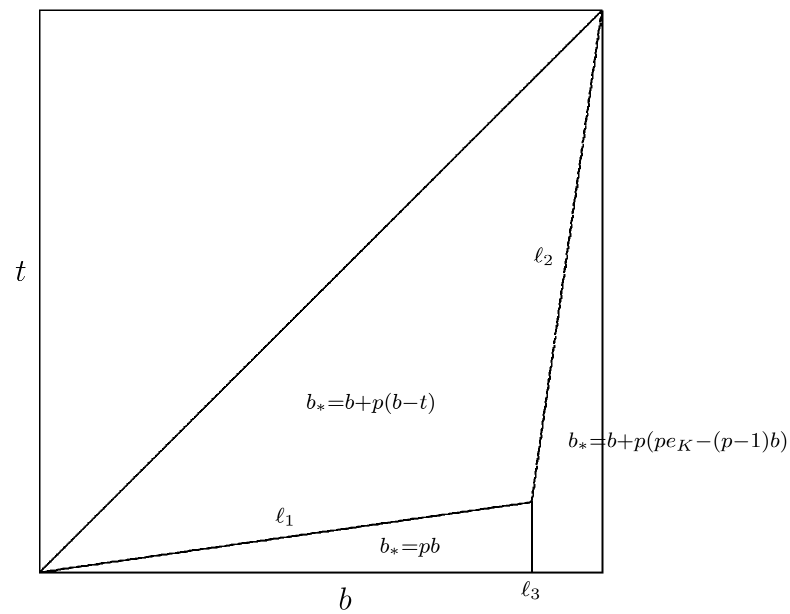
2.2.3 Strong twists alter ramification breaks

Let $\bar{G} = \mathrm{Gal}(\bar{K}/K)$ denote the absolute Galois group. We will call the fixed field of the kernel of a representation of $\bar{G}$, the fixed field of the representation. Let χ_x, χ_{xy}, χ_z be 1-dimensional characters with fixed fields $K(x)$, $K(xy)$ and $K(z)$ respectively. Let V denote the 2-dimensional representation of $\bar{G}$ with character $\chi_y + \chi_{xy}$ and fixed field $N_1 = K(x,y)$. Then $N_z = K(x, yz)$ is the fixed field of the twisted representation $V \otimes \chi_z$. The 'strength' of the twist by χ_z is parametrized by t, the ramification break of $K(z)/K$.

Consider the following diagram with the formula for b_* displayed as a function of (b,t) in each of three relevant regions that lie below the diagonal line $t = b$. The boundaries of these regions are: the line $t = b$; the segment ℓ_1, which is on

the line $t = b/p$; the segment ℓ_2, which is on $t = p(b - e_K)$; and the segment ℓ_3, which is on $b = p^2 e_K/(p^2 - 1)$.

Diagram: b_* as a function of (b, t).



Now view N_z is a twist of N_1 and observe that 'strong' twists change ramification filtrations, while 'weak' twists preserve them: If the twist is 'weak' and thus t is relatively small ($t < b/p$ or $t < p(b - e_K)$), the formula for b_* in N_z is the same as in N_1. Otherwise the formulas for b_* are different (although if $t < b$, N_z/K still has only one ramification break). If we strengthen our twist further and choose $t > b$, then N_z/K will have two ramification breaks.

Why is this so? Why are the values of the second refined breaks in N_z and N_1 equal when $t < b/p$ or $t < p(b - e_K)$? Observe that the formula for b_* results from the expression for $v_L(\eta_z)$ determined in Proposition 10. Note furthermore that the proof of Proposition 10 describes η_z completely in terms of r_z. So our question becomes: Why do $r_z, r_1 \in L$ "agree" under $t < b/p$ or $t < p(b - e_K)$? When $t < b/p$, because they are equal. Recall (1). So where it matters, the twist has no effect! Now consider $t < p(b - e_K)$ with $t > b/p$. Motivated by our answer for $t < b/p$, observe that $t < p(b - e_K)$ is equivalent to $v_L(\beta) < v_L(\delta')$, where δ' was defined in Lemma 8. Returning to (1), we conclude that they "agree" because they are equivalent, $r_z \equiv r_1 \bmod \beta\mathfrak{P}_L$.

Proof (Theorem 5) Recall that N/K is a fully ramified, bicyclic extension with one ramification break at b. If $\zeta \in K$ and so the pth roots of unity are present, the result is contained in Proposition 10 and Corollary 11. To apply these results when $\zeta \notin K$, we consider the related Kummer extension $N(\zeta)/K(\zeta)$ with $d = [K(\zeta) : K]$. By abuse of notation use σ, γ to represent automorphisms in $\mathrm{Gal}(N(\zeta)/K)$, so that $\langle \sigma, \gamma \rangle = \mathrm{Gal}(N(\zeta)/K(\zeta)) = \mathrm{Gal}(N/K)$. Pick any $\rho_0 \in N$ with $v_N(\rho_0) = b$. Then $v_{N(\zeta)}(\rho_0) = db$. Using the Herbrand function (Ser79, IV §3), the ramification break of $N(\zeta)/K(\zeta)$ is db. Recall from the beginning of §2, that ω is defined to be the unique $p^f - 1$ root of unity such that $(\gamma - 1)\rho_0 \equiv -\omega(\sigma - 1)\rho_0 \bmod \mathfrak{P}_N^{2b+1}$. Since $(\gamma\sigma^{[\omega]} - 1)\rho_0 \equiv 0 \bmod \rho_0^2\pi_N$ in N, $(\gamma\sigma^{[\omega]} - 1)\rho_0 \equiv 0 \bmod \rho_0^2\pi_{N(\zeta)}$ in $N(\zeta)$. Therefore the ω defined here is the same as the ω defined in §2.2.2 for $N(\zeta)/K(\zeta)$. And $v_{N(\zeta)}((\Theta - 1)\rho_0) = db + db_*$, where db_* is determined by Proposition 10 with b replaced by db and $e_{K(\zeta)} = de_K$. The result follows now after the integer d is removed everywhere. $\square$

3 Galois Module Structure in Bicyclic Extensions

We are interested in the relevance of the second refined break b_* for Galois module structure. Let N/K be a fully ramified, bicyclic extension with one break b in its ramification filtration and assume the notation of §2.

In Theorem 12 of §3.1, we determine just enough of the $\mathbb{F}_q[G]$-structure of $\mathfrak{P}_N^r/p\mathfrak{P}_N^r$ to prove that, if $b_* < (p - 1 + 1/p)b$, this structure depends upon b_*. As a result, the $\mathfrak{O}_T[G]$-structure of ideals also depends upon b_*.

Next, because it is easily done, we assume in §3.2 that we have maximal refined ramification $b_* = pb$, and in Theorem 18 explicitly describe, in a transparent way, the $\mathfrak{O}_T[G]$-structure of each ideal $\mathfrak{P}_N^r$.

Based upon (BE02), we conjecture that the our result concerning the relevance of b_* is sharp – namely, that the $\mathfrak{O}_T[G]$-structure of each ideal $\mathfrak{P}_N^r$ under $(p - 1 + 1/p)b < b_* < pb$, which we call *near maximal refined ramification*, is independent of b_* and in fact agrees with the structure given in Theorem 18.

3.1 On modular Galois module structure

Identify $\Theta \in \mathfrak{O}_T[G]$ with its image in $\mathbb{F}_q[G]$, and observe that $(\Theta)^p = 1$ in $\mathbb{F}_q[G]$. There are exactly p indecomposable modules over $\mathbb{F}_q[\Theta]$, namely

$L(i) = \mathbb{F}_q[x]/(x-1)^i$ for $1 \le i \le p$, where Θ acts via multiplication by x. This means that $\mathfrak{P}_N^r/p\mathfrak{P}_N^r$ is uniquely expressible as

$$\mathfrak{P}_N^r/p\mathfrak{P}_N^r \cong \bigoplus_{i=1}^{p} L(i)^{a_i}$$

for some integers $a_i \ge 0$. Here we determine a_p, and in particular find

Theorem 12

$$a_p = \dim_{\mathbb{F}_q}\left((\Theta-1)^{p-1}\mathfrak{P}_N^r/p\mathfrak{P}_N^r\right) =$$
$$pe_K + \left\lceil \frac{r}{p} \right\rceil - \left\lceil \frac{r-b}{p} \right\rceil - b - \begin{cases} \frac{b_*-b}{p} & \textit{for } b_* < (p-1+1/p)b, \\ b + \left\lceil \frac{r-pb}{p^2} \right\rceil - \left\lceil \frac{r+(p-1)b}{p^2} \right\rceil & \textit{otherwise.} \end{cases}$$

This result for $b_* = pb$ follows from Theorem 18. In this section, we verify it for $b_* < pb$, which allows us to use the fact that $c = (b_* - b)/p$ is an integer.

We begin by establishing an $\mathfrak{O}_T$-basis for $\mathfrak{P}_N^r$, a basis that will also serve as a $\mathbb{F}_q$-basis for

$$\mathcal{M} = \mathfrak{P}_N^r/p\mathfrak{P}_N^r.$$

Let $\rho_m \in N$ be any element with $v_N(\rho_m) = b + pm$ and observe that since $b_* \equiv b \bmod p$ and $\gcd(b,p) = 1$, $\{v_N((\Theta-1)^i p\rho_m) : i = 0, \dots, p-1\}$ is a complete set of residues modulo p. As m varies over $\mathbb{Z}$, the resulting elements $(\Theta-1)^i p\rho_m$ will lie in one-to-one correspondence, via valuation v_N, with $\mathbb{Z}$. Collect those with $r \le v_N((\Theta-1)^i p\rho_m) \le r + p^2 e_K - 1$. We have a $\mathfrak{O}_T$-basis for $\mathfrak{P}_N^r$. So that we can follow the effect of Θ upon this basis, we will replace certain ρ_m with ρ_m^* of equal valuation. This is done in Lemma 14. But first we require a technical lemma.

Lemma 13 *For any $\omega \in \mathfrak{O}_T$, we have the congruence in $\mathfrak{O}_T[\sigma]$*

$$(\sigma^{[\omega]})^p - 1 \equiv (w - w^p) \sum_{i=1}^{p-1} \binom{p}{i} (\sigma-1)^i \bmod p^2\mathfrak{O}_T[\sigma].$$

Recall $\Theta = \gamma\sigma^{[\omega]} \in \mathfrak{O}_T[G]$. Then there is a unit $u(\sigma) \in \mathfrak{O}_T[\sigma]^*$ defined by

$$(\Theta-1)^p + \sum_{i=1}^{p-1} \binom{p}{i} (\Theta-1)^i = p(\sigma-1)u(\sigma),$$

satisfying

$$u(\sigma) \equiv (\omega - \omega^p) \sum_{i=1}^{p-1} \left[\frac{1}{p}\binom{p}{i}\right] (\sigma-1)^{i-1} \equiv (\omega - \omega^p) \sum_{i=1}^{p-1} (-1)^{i+1} i^{-1} (\sigma-1)^{i-1}$$

modulo $p\mathfrak{O}_T[\sigma]$. *In particular,* $(\Theta-1)^p = [u(\sigma)(\sigma-1) - w(\Theta)(\Theta-1)]\cdot p$ *where* $u(\sigma)$ *and* $w(\Theta) = \sum_{s=1}^{p-1} p^{-1}\binom{p}{s}(\Theta-1)^{s-1}$ *are both units in* $\mathfrak{O}_T[G]$.

PROOF. We work initially in the truncated polynomial ring $\mathbb{Q}[W, F]/(F^{2p})$. In this ring we have the (finite) binomial expansion $(1+F)^W = \sum_{i=0}^{2p-1}\binom{W}{i}F^i = \sum_{i=0}^{p-1}\binom{W}{i}F^i + \sum_{i=0}^{p-1}\binom{W}{p+i}F^{p+i}$. Now, as observed in the proof of (BE05, Lem 2.2), for $0 \le i \le p-1$ we have $p\binom{W}{p+i} \in \mathbb{Z}_{(p)}[W]$, $p\binom{W}{p+i} \equiv (W - W^p)\binom{W}{i}$ (mod $p\mathbb{Z}_{(p)}[W]$). Hence there is a polynomial $e(F, W) \in \mathbb{Z}_{(p)}[W, F]$ such that

$$\begin{aligned}(1+F)^W =& \sum_{i=0}^{p-1}\binom{W}{i}F^i + \frac{F^p}{p}\left(\sum_{i=0}^{p-1}(W-W^p)\binom{W}{i}F^i + p\cdot e(F,W)\right)\\ =& (1+F)^{[W]}\left(1+(W-W^p)\frac{F^p}{p}\right) + F^p e(W,F).\end{aligned}$$

Raising both sides to the power p, using $(F^p)^2 = 0$, and observing that $((1+F)^W)^p = ((1+F)^p)^W$ by properties of (infinite) binomial series, we obtain the following identity in $\mathbb{Z}_{(p)}[W, F]/(F^{2p})$:

$$\Big((1+F)^p\Big)^W = \Big((1+F)^{[W]}\Big)^p\Big(1+(W-W^p)F^p\Big) + p\Big((1+F)^{[W]}\Big)^{p-1}F^p e(W,F).$$

Consider its image under the homomorphism from $\mathbb{Z}_{(p)}[W, F]/(F^{2p})$ to $R = (\mathfrak{O}_T/p^2\mathfrak{O}_T)[\sigma]$ which takes W to ω and F to $f = \sigma - 1$. This homomorphism is well-defined because $f^p = -\sum_{i=1}^{p-1}\binom{p}{i}f^i \in pR$, so that $f^{2p} \in p^2R$. In R this identity becomes

$$1 = (\sigma^{[\omega]})^p\Big(1+(\omega-\omega^p)f^p\Big).$$

Note that $(1+(\omega-\omega^p)f^p)(1-(\omega-\omega^p)f^p) = 1$ in R, and so $(\sigma^{[\omega]})^p = 1-(\omega-\omega^p)f^p$ in R. Moreover expanding $\sigma^p = (1+f)^p$ using the binomial theorem yields $f^p = -\sum_{i=1}^{p-1}\binom{p}{i}f^i$, and thus $(\sigma^{[\omega]})^p - 1 = (\omega-\omega^p)\sum_{i=1}^{p-1}\binom{p}{i}f^i$ in R.

Use the binomial expansion $(\sigma^{[\omega]})^p = \Theta^p = \Big((\Theta-1)+1\Big)^p = (\Theta-1)^p + 1 + \sum_{i=1}^{p-1}\binom{p}{i}(\Theta-1)^i$, and we obtain the statements concerning $u(\sigma)$. $\square$

Lemma 14 *There are* $\rho_m, \rho_m^* \in N$ *with* $v_N(\rho_m) = v_N(\rho_m^*) = b + pm$ *satisfying*

$$\begin{aligned}(\Theta-1)p\rho_m - (\Theta-1)^p\rho_{m+c}^* - \sum_{i=1}^{p-1}\left[\frac{1}{p}\binom{p}{i}\right](\Theta-1)^i p\rho_{m+c}^*\\ = \begin{cases} p\rho_{m+b} & \textit{for } m \not\equiv -b \bmod p\\ p\rho_{m+pe_K-(p-2)b} & \textit{for } m \equiv -b \bmod p\end{cases}\end{aligned}$$

PROOF. For $0 \leq k < e_K$ choose $\alpha_k \in L$ with $v_L(\alpha_k) = b + pk$. Since $u_\gamma := \sum_{i=1}^{p-1} \left[\frac{1}{p}\binom{p}{i}\right] (\gamma-1)^{i-1} \in \mathbb{Z}_p[\gamma]^*$, $v_L(u_\gamma^m \alpha_k) = b + pk$ for all $m \in \mathbb{Z}$. Let $\alpha_{k+me_K} = (-pu_\gamma)^m \alpha_k$. So $v_L(\alpha_k) = b + pk$ for all $k \in \mathbb{Z}$ and $\alpha_{k+e_K} = -\sum_{i=1}^{p-1} \binom{p}{i}(\gamma-1)^{i-1}\alpha_k$. As a result, $(\gamma-1)\alpha_{k+e_K} = (\gamma-1)^p\alpha_k$, because $(\gamma-1)^p = -\sum_{i=1}^{p-1}\binom{p}{i}(\gamma-1)^i$.

Now $v_L((\gamma-1)^i\alpha_k) = (i+1)b + pk$ for $0 \leq i \leq p-1$. Use (Ser79, V §3 Lem 4) to find $\mu_{i,k} \in N$ with $v_N(\mu_{i,k}) = (1+pi)b + p^2k$ and $\Phi_p(\sigma)\mu_{i,k} = (\gamma-1)^i\alpha_k$. Since $\Phi_p(\sigma)(\Theta-1) = (\gamma-1)\Phi_p(\sigma)$, $\Phi_p(\sigma)\cdot((\Theta-1)\mu_{i,k} - \mu_{i+1,k}) = 0$ for $0 \leq i \leq p-2$. Also $\Phi_p(\sigma)\cdot((\Theta-1)\mu_{p-1,k} - \mu_{1,k+e_K}) = 0$.

By the Normal Basis Theorem, if $\Phi_p(\sigma)\nu = 0$ for $\nu \in N$ with $v_N(\nu) \not\equiv b \bmod p$, then there is a $\theta \in N$ with $v_N(\theta) = v_N(\nu) - b$ and $(\sigma-1)\theta = \nu$. Recall $u(\sigma) \in \mathfrak{O}_T[\sigma]^*$ defined in Lemma 13, and use the Normal Basis Theorem to find $\rho_s^* \in N$ with $v_N(\rho_s^*) = b + ps$ such that

$$(\Theta-1)\mu_{i,k} = (\sigma-1)u(\sigma)\rho^*_{ib+pk+c} + \begin{cases} \mu_{i+1,k} & \text{for } 0 \leq i < p-1 \\ \mu_{1,k+e_K} & \text{for } i = p-1 \end{cases}$$

Now define $\rho_s \in N$ with $v_N(\rho_s) = b + ps$ by $\rho_{bi+pk} = \mu_{i,k}$. And use Lemma 13 to replace $(\sigma-1)u(\sigma)\rho_s^*$ by $(1/p)\cdot((\Theta-1)^p\rho_s^* + \sum_{i=1}^{p-1}\binom{p}{i}(\Theta-1)^i\rho_s^*)$. □

PROOF (Theorem 12 when $b_* < pb$) We have an $\mathfrak{O}_T$-basis for $\mathfrak{P}_N^r$ consisting of the elements

$$p\rho_m, (\Theta-1)\rho_m, (\Theta-1)^2p\rho_m, \ldots, (\Theta-1)^{p-1}p\rho_m$$

for

$$\frac{r-b}{p} - pe_K \leq m < \frac{r}{p} + c - b_* \qquad \text{(Range A);}$$

and

$$(\Theta-1)^{j_m+1}\rho_m^*, \ldots, (\Theta-1)^{p-1}\rho_m^*, p\rho_m, \ldots, (\Theta-1)^{j_m}p\rho_m$$

for

$$\frac{r}{p} + c - b_* \leq m < \frac{r-b}{p} \qquad \text{(Range B).}$$

Here m is restricted to integer values, and $j_m \in \{0, \ldots, p-2\}$ is such that $(\Theta-1)^{j_m+1}\rho_m^* \in \mathfrak{P}_N^r$ but $(\Theta-1)^{j_m}\rho_m^* \notin \mathfrak{P}_N^r$.

Clearly $\mathcal{M}$ is spanned over $\mathbb{F}_q$ by (the images of) these $\mathfrak{O}_T$-basis elements. So $\mathcal{M}$ is spanned over $\mathbb{F}_q[\Theta]$ by the $p\rho_m$ for $m \in$ Range A, along with the $(\Theta-1)^{j_m+1}\rho_m^*$, and the $p\rho_m$ for $m \in$ Range B. We want to determine the $\mathbb{F}_q$-dimension of $(\Theta-1)^{p-1}\mathcal{M}$. So apply $(\Theta-1)^{p-1}$. Since the image of $(\Theta -$

$1)^{p-1}p\rho_m$ in $\mathcal{M}$ is clearly zero for $m \in$ Range B, we are left with an $\mathbb{F}_q$-generating set for $(\Theta-1)^{p-1}\mathcal{M}$ consisting of

$$(\Theta-1)^{p-1}p\rho_m \text{ for } m \in \text{Range A, and } (\Theta-1)^{j_m+p}\rho_m^* \text{ for } m \in \text{Range B.} \quad (3)$$

This set of e_K elements is not a basis. Using the relationships in Lemma 14, we should be able to replace certain $(\Theta-1)^{p-1}p\rho_m$ with $(\Theta-1)^{j_m+p}\rho_{m'}^*$ for some $m' \in$ Range B, or eliminate it entirely. Of course, since the relationships in Lemma 14 are the only "extra" relations, once we have made all such replacements/eliminations, we will be left with a $\mathbb{F}_q$-basis for $(\Theta-1)^{p-1}\mathcal{M}$.

Split Range B into a disjoint union of sets Range $\mathrm{B}_0, \dots,$ Range B_{p-2} where Range B_j consists of those m with $j_m = j$. In other words, Range $B_j = \{m \in \mathbb{Z} : r-(j+1)b_* - b \le pm < r - jb_* - b\}$.

Take the relationship in Lemma 14, replace m with $m-c$ and multiply it by $(\Theta-1)^{p-2}$. We are interested in the situation where $m-c \in$ Range A and $m \in$ Range B. Since the "length" of Range B_{p-2} is $b_*/p > c$, this actually occurs when $m-c \in$ Range A and $m \in$ Range B_{p-2}. Since $(\Theta-1)^{i+p-2}p\rho_m^* \in p\mathfrak{P}_N^r$ for $i \ge 1$ and $m \in$ Range B_{p-2}, the relationship in Lemma 14 simplifies to

$$(\Theta-1)^{2p-2}\rho_m^* = (\Theta-1)^{p-1}p\rho_{m-c} - (\Theta-1)^{p-2}p\rho_{f(m)-c}, \quad (4)$$

where

$$f(m) = m + \begin{cases} b & \text{if } m \not\equiv c-b \bmod p, \\ pe_K - (p-2)b & \text{if } m \equiv c-b \bmod p. \end{cases}$$

It is helpful, since we are interested in other relationships similar to (4), to observe that in general,

$$v_N((\Theta-1)^{j+p}\rho_m^*) = v_N((\Theta-1)^{j+1}p\rho_{m-c}) < v_N((\Theta-1)^{j}p\rho_{f(m)-c)}.$$

So in regards to (4) where $j=p-2$, $(\Theta-1)^{p-2}p\rho_{f(m)-c}$ should be regarded as "error." We can remove $(\Theta-1)^{p-1}p\rho_{m-c}$ from the set of generators (3) for those $m-c \in$ Range A such that $m \in$ Range B_{p-2} and $(\Theta-1)^{p-2}p\rho_{f(m)-c} \in p\mathfrak{P}_N^r$. But before we do so, need to consider $(\Theta-1)^{p-2}p\rho_{f(m)-c} \notin p\mathfrak{P}_N^r$. Indeed we will find that we do not need to treat these cases separately.

Notice that $f(m)-m \ge b > b_*/p$, which is the approximate "length" of each Range B_j. So for $m \in$ Range B_{p-2}, it is certainly the case that $f(m) \in$ Range B_k for some $k \le p-3$. Moreover if we denote iteration in the usual way, $f^2 = f \circ f$, $f^3 = f \circ f \circ f$, etc., it will be the case that $f^2(m) \in$ Range B_k for some $k \le p-4$ and so on. Think of Range B_{-1} as including those m such that $p\rho_m^* \in p\mathfrak{P}_N^r$ and so are zero in $\mathcal{M}$.

Take the relationship in Lemma 14, replace m with $m-c$ and multiply by $(\Theta-1)^j$. For $m \in$ Range B_k with $k \le j$, we have $(\Theta-1)^{i+j}p\rho_m^* \in p\mathfrak{P}_N^r$ for

$i \geq 1$. And so the relationship simplifies to $(\Theta-1)^{j+p}\rho^*_m = (\Theta-1)^{j+1}p\rho_{m-c} - (\Theta-1)^j p\rho_{f(m)-c}$. As a result, in addition to (4), we also have

$$\begin{aligned}
(\Theta-1)^{2p-3}\rho^*_{f(m)} &= (\Theta-1)^{p-2}p\rho_{f(m)-c} - (\Theta-1)^{p-3}p\rho_{f^2(m)-c},\\
(\Theta-1)^{2p-4}\rho^*_{f^2(m)} &= (\Theta-1)^{p-3}p\rho_{f^2(m)-c} - (\Theta-1)^{p-4}p\rho_{f^3(m)-c},\\
&\vdots\\
(\Theta-1)^{p}\rho^*_{f^{p-2}(m)} &= (\Theta-1)p\rho_{f^{p-2}(m)-c} - p\rho_{f^{p-1}(m)-c}.
\end{aligned}$$

As a result, for $m-c \in$ Range A and $m \in$ Range B_{p-2} we have

$$\sum_{j=0}^{p-2}(\Theta-1)^{p+j}\rho^*_{f^{p-j-2}(m)} = (\Theta-1)^{p-1}p\rho_{m-c} - p\rho_{f^{p-1}(m)-c},$$

where for $j < p-2$ either $f^{p-j-2}(m) \in$ Range B_j and $(\Theta-1)^{p+j}\rho^*_{f^{p-j-2}(m)}$ is a nontrivial generator listed in (3), or $(\Theta-1)^{p+j}\rho^*_{f^{p-j-2}(m)} = 0$ in $\mathcal{M}$. In any case, $(\Theta-1)^{p-1}p\rho_{m-c}$ is clearly expressed in terms of other generators and can be removed *if and only if* $p\rho_{f^{p-1}(m)-c} \in p\mathfrak{P}^r_N$. Notice that

$$f^{p-1}(m) = m + \begin{cases}(p-1)b & \text{if } m \equiv c \bmod p,\\ pe_K & \text{if } m \not\equiv c \bmod p.\end{cases}$$

As a result, we can remove those elements $(\Theta-1)^{p-1}p\rho_{m-c}$ for $m \in$ Range B_{p-2}, namely

$$\frac{r}{p} + c - b_* \leq m < \frac{r+b}{p} + 2c - b_*$$

such that $pf^{p-1}(m) \geq r + b_* - 2b$. Once we have done so, we will have an $\mathbb{F}_q$-basis for $(\Theta-1)^{p-1}\mathcal{M}$.

Since $f^{p-1}(m) = m + pe_K$ for $m \not\equiv c \bmod p$ we can remove all $m \not\equiv c \bmod p$. We can also remove all $m \equiv c \bmod p$ if $(p^2-p+1)b \geq pb_*$. Doing so and keeping track of how many elements were removed yields part of the statement of Theorem 12. To get the statement under near maximal refined ramification, notice that we need to "put back" one element for each integer $m \equiv c \bmod p$, that satisfies $r/p + c - b_* \leq m$ and $pf^{p-1}(m) < r + b_* - 2b$. $\square$

We now state two corollaries of Theorem 12.

Corollary 15 *Let K be a finite extension of $\mathbb{Q}_p$ and let N_1, N_2 be two fully ramified bicyclic extensions with unique ramification break number b. Assume that the two second refined ramification breaks satisfy $b^{(1)}_*, b^{(2)}_* < (p-1+1/p)b$. If $b^{(1)}_* \neq b^{(2)}_*$, then for each r, $\mathfrak{P}^r_{N_1} \not\cong \mathfrak{P}^r_{N_2}$ as $\mathfrak{O}_T[G]$-modules.*

Motivated by the diagram in §2.2.3, we observe that when the break number b is large enough, the hypothesis on the second refined ramification numbers can be replace with a hypothesis on b.

Corollary 16 *Let K be a finite extension of $\mathbb{Q}_p$ and let N_1, N_2 be two fully ramified bicyclic extensions with unique ramification break number b satisfying*

$$\left(1 - \frac{p^2 - 2p + 1}{p^3 - 2p + 1}\right) \cdot \frac{pe_K}{p-1} < b < \frac{pe_K}{p-1}.$$

If the two second refined ramification breaks are different, $b_*^{(1)} \neq b_*^{(2)}$, then for each r, $\mathfrak{P}_{N_1}^r \not\cong \mathfrak{P}_{N_2}^r$ as $\mathfrak{O}_T[G]$-modules.

3.2 Maximal refined ramification and Galois module structure

In this section we assume $b_* = pb$ and establish an explicit integral basis for $\mathfrak{P}_N^r$ over $\mathfrak{O}_T$ upon which we can follow the Galois action in a particularly transparent way.

Recall the notation of §2, in particular $\Theta = \gamma\sigma^{[\omega]}$. Using Lemma 3, $b_* = pb$ means that $v_N((\Theta - 1)\rho = v_N(\rho) + pb$ for $\rho \in N$ if $v_N(\rho) \equiv b \bmod p$ but $v_N(\rho) \not\equiv (1-p)b \bmod p^2$. As a result, given any $\rho_m \in N$ with $v_N(\rho_m) = b + p^2 m$, we have $v_N((\Theta - 1)^i \rho_m) = (1 + ip)b + p^2 m$ for $0 \leq i \leq p-1$ and thus

$$\rho_m^{(i,j)} := (\Theta - 1)^i (\sigma - 1)^j \rho_m \text{ satisfies } v_N(\rho_m^{(i,j)}) = (1 + j + ip)b + p^2 m \qquad (5)$$

for $0 \leq i, j \leq p-1$. Since $\{v_N(\rho_m^{(i,j)}) : 0 \leq i, j \leq p-1\}$ is a complete set of residues modulo p^2, $\{v_N(\rho_m^{(i,j)}) : 0 \leq i, j \leq p-1, m \in \mathbb{Z}\} = \mathbb{Z}$ and since N/T is fully ramified, we can use the $\rho_m^{(i,j)}$ to construct an $\mathfrak{O}_T$-basis for $\mathfrak{P}_N^r$. For example, simply choose $\rho_m^{(i,j)}$ with $r \leq v_N(\rho_m^{(i,j)}) \leq r + p^2 e_K - 1$.

So that the Galois action can be followed on this basis, we must modify this construction (but only slightly). Consider the 'exponent' (i, j) to be a two digit p-ary integer $ip + j$. The larger the integer (i, j) then, the larger the valuation $v_N(\rho_m^{(i,j)})$. Furthermore recall the diagram in §2.2.3. Since $b_* = pb$ we have $b < p^2 e_K/(p^2 - 1)$. So there are values of m such that $r \leq v_N(p\rho_m^{(0,0)}) < v_N(p\rho_m^{(p-1,p-1)}) < r + p^2 e_K$. For these m set $(i, j)_m = (p-1, p-1)$. Otherwise $r \leq v_N(\rho_m^{(p-1,p-1)}) < v_N(p\rho_m^{(0,0)}) < r + p^2 e_K$. For each of these other values of m, let $(i, j)_m$ be the p-ary integer such that $r \leq v_N(\rho_m^{(i,j)_m + (0,1)}) \leq v_N(\rho_m^{(p-1,p-1)}) < v_N(p\rho_m^{(0,0)}) \leq v_N(p\rho_m^{(i,j)_m}) < r + p^2 e_K$. For each integer m such that $(r-b)/p^2 - e_K \leq m < (r-b)/p^2$ define

$$\mathcal{M}(m) = \mathfrak{O}_T \rho_m^{(i,j)_m + (0,1)} + \cdots + \mathfrak{O}_T \rho_m^{(p-1,p-1)} + \mathfrak{O}_T p\rho_m^{(0,0)} + \cdots + \mathfrak{O}_T p\rho_m^{(i,j)_m}.$$

Note that when $(i,j)_m = (p-1,p-1)$ we consider the sum $\mathfrak{O}_T\rho_m^{(i,j)_m+(0,1)} + \cdots + \mathfrak{O}_T\rho_m^{(p-1,p-1)}$ to be empty. In other words, $(p-1,p-1)+(0,1)$ should be considered larger than $(p-1,p-1)$. We find

$$\mathfrak{P}_N^r = \sum_{m=A_r-e_K}^{A_r-1} \mathcal{M}(m) \text{ where } A_r = \left\lceil \frac{r-b}{p^2} \right\rceil$$

and $\lceil \cdot \rceil$ denotes the least integer function (ceiling function).

The $\mathfrak{O}_T[G]$-structure of $\mathfrak{P}_N^r$, namely Theorem 18, follows then from the following lemma and some basic combinatorics.

Lemma 17 *Each $\mathcal{M}(m)$ is isomorphic to an ideal of $\mathfrak{O}_T[G]$. Indeed, if we write $(i,j)_m$ as (i_m, j_m), then*

$$\mathcal{M}(m) \cong \left\langle p, (\Theta-1)^{i_m}(\sigma-1)^{j_m+1}, (\Theta-1)^{i_m+1} \right\rangle$$

PROOF. Let $\phi_{(i,j)} = (\Theta-1)^i(\sigma-1)^j$. So $\rho_m^{(i,j)} = \phi_{(i,j)}\rho_m$. Now list the $\mathfrak{O}_T$-basis elements of $\mathcal{M}(m)$ in an array, dropping the ρ_m from each element:

$$\begin{array}{cccc|ccc}
p\phi_{(0,0)}, & \cdots & & p\phi_{(i_m,0)}, & \phi_{(i_m+1,0)}, & \cdots & \phi_{(p-1,0)}, \\
 & \ddots & & & & \ddots & \\
p\phi_{(0,j_m)}, & \cdots & & p\phi_{(i_m,j_m)}, & \phi_{(i_m+1,j_m)}, & \cdots & \phi_{(p-1,j_m)}, \\
\cline{3-4}
p\phi_{(0,j_m+1)}, & \cdots & \multicolumn{1}{|c}{} & \phi_{(i_m,j_m+1)}, & \phi_{(i_m+1,j_m+1)}, & \cdots & \phi_{(p-1,j_m+1)}, \\
 & \ddots & \multicolumn{1}{|c}{} & & & \ddots & \\
p\phi_{(0,p-1)}, & \cdots & \multicolumn{1}{|c}{} & \phi_{(i_m,p-1)}, & \phi_{(i_m+1,p-1)}, & \cdots & \phi_{(p-1,p-1)}.
\end{array} \tag{6}$$

The boundary between elements in and out of $p\mathfrak{O}_T[G]$ is marked. We would like to show that the $\mathfrak{O}_T$-span of (6) is the $\mathfrak{O}_T[G]$-ideal generated by $p\phi_{(0,0)}$, $\phi_{(i_m,j_m+1)}$, $\phi_{(i_m+1,0)}$. But this is clear, once we know that this $\mathfrak{O}_T$-span is closed under σ and Θ, and this follows from the fact that $(\sigma-1)^p$ and $(\Theta-1)^p \in p\mathfrak{O}_T[G]$. $\square$

Theorem 18 *Let $b_* = pb$. For $0 \le s \le p^2-1$ let (i_s, j_s) denote the p-ary expansion of s. So $s = i_s p + j_s$. For $0 \le s \le p^2-1$, let $\mathcal{I}_s$ be the $\mathfrak{O}_T[G]$ ideal $\langle p, (\Theta-1)^{i_s}(\sigma-1)^{j_s+1}, (\Theta-1)^{i_s+1}\rangle$, with $\mathcal{I}_{p^2-1} = p\mathfrak{O}_T[G] \cong \mathfrak{O}_T[G]$. Then*

$$\mathfrak{P}_N^r \cong \bigoplus_{s=0}^{p^2-2} \mathcal{I}_s^{\left\lceil \frac{r-(s+1)b}{p^2} \right\rceil - \left\lceil \frac{r-(s+2)b}{p^2} \right\rceil} \oplus \mathcal{I}_{p^2-1}^{e_K - b + \left\lceil \frac{r}{p^2} \right\rceil - \left\lceil \frac{r-b}{p^2} \right\rceil}$$

as $\mathfrak{O}_T[G]$-modules. This structure is parametrized by r, ω, p, e_K and b.

PROOF (Theorem 12 when $b_*=pb$). Each $\mathfrak{O}_T[G]$-module $\mathcal{I}_s$ has an $\mathfrak{O}_T$-basis as in (6). To compute $\dim_{\mathbb{F}_q}((\Theta-1)^{p-1}\mathcal{I}_s/p\mathcal{I}_s)$, simply apply $(\Theta-1)^{p-1}$ to each of these $\mathfrak{O}_T$-basis elements. By Lemma 13, for $i\geq 1$ we have $(\Theta-1)^{p-1}\cdot(\Theta-1)^i(\sigma-1)^j\equiv u(\sigma)\cdot(\Theta-1)^{i-1}(\sigma-1)^{j+1}p \bmod p(\Theta-1)^i(\sigma-1)^j$, where $u(\sigma)$ is a unit. As a result,

$$\dim_{\mathbb{F}_q}((\Theta-1)^{p-1}\mathcal{I}_s/p\mathcal{I}_s)=\begin{cases}p-1 & 0\leq s\leq p-1,\\ p-2 & p\leq s\leq p^2-p-1, s\not\equiv -1 \bmod p,\\ p-1 & p\leq s\leq p^2-p-1, s\equiv -1 \bmod p,\\ p-1 & p^2-p\leq s\leq p^2-2,\\ p & s=p^2-1.\end{cases}$$

Let m_s denote the multiplicity of $\mathcal{I}_s$ in the statement of Theorem 18. So $m_s=\lceil (r-(s+1)b)/p^2\rceil-\lceil (r-(s+2)b)/p^2\rceil$ for $0\leq s\leq p^2-2$. Therefore because $p-1=(p-2)+1$, we have $\dim_{\mathbb{F}_q}((\Theta-1)^{p-1}\mathfrak{P}_N^r/p\mathfrak{P}_N^r)=pm_{p^2-1}+(p-2)\sum_{s=0}^{p^2-2}m_s+\sum_{s=0}^{p-1}m_s+\sum_{s=p^2-p}^{p^2-2}m_s+\sum_{k=2}^{p-1}m_{kp-1}$. These are for the most part telescoping sums, and so the expression simplifies to $\dim_{\mathbb{F}_q}((\Theta-1)^{p-1}\mathfrak{P}_N^r/p\mathfrak{P}_N^r)=pe_k-2b+\lceil r/p^2\rceil-\lceil (r-b)/p^2\rceil-\lceil (r-(p+1)b)/p^2\rceil+\lceil (r+(p-1)b)/p^2\rceil+\sum_{k=2}^{p-1}m_{kp-1}$. It remains to recognize that $\sum_{k=0}^{p-1}m_{kp-1}=\lceil r/p\rceil-\lceil (r-b)/p\rceil$, which follows from the fact that both count the number of integers i such that $(r-b)/p\leq i\leq (r-1)/p$. Each term m_{kp-1} in the sum simply counts those integers $\equiv kb \bmod p$.

4 Conclusion

This paper grows out of the on-going effort to generalize the biquadratic results of (BE02) to $p>2$. Thus far, several themes have emerged and a number of questions have been raised, all of which which bear repeating.

The central theme is the role of truncated exponentiation. Its appearance within the group ring $\mathfrak{O}_T[G]$ for G elementary abelian has led to the refined ramification filtration (BE05). Notably, the definition of refined ramification break numbers remains tied to a choice of element and so the refined breaks (beyond the first two) cannot, as yet, be said to be canonical. In addition, it has been observed in the context of quaternion extensions (EH07) that the refined ramification filtration has some influence on breaks in the usual ramification filtration. And so there is much remaining work to determine if/how these two filtrations fit together.

The appearance of truncated exponentiation among the generators of the extension in §2.2.2 as well as the notion of maximal refined ramification and the ease and transparency in §3.2 are the motivation for (Eld). This, along any

connection with Artin-Hasse exponentiation and explicit reciprocity (FV02), warrant further investigation.

In (BE02, §4) and then here in §2.2.3, a question is raised concerning how twists by characters of Galois representations effect ramification, refined ramification and Galois module structure. One consequence of this question is the suggestion that the problem of Galois module structure be broken in two: (1) The determination of nice classes of extension, for which the Galois module structure can be easily determined. *e.g.* (Eld) (2) The problem of Galois module structure under twisting, which remains very much open.

References

[BE] Nigel P. Byott and G. Griffith Elder, *A valuation criterion for normal bases in elementary abelian extensions*, to appear in the Bull. London Math. Soc.

[BE02] ———, *Biquadratic extensions with one break*, Can. Math. Bull. **45** (2002), no. 2, 168–179.

[BE05] ———, *New ramification breaks and additive Galois structure*, J. Théor. Nombres Bordeaux **17** (2005), no. 1, 87–107.

[CR90] Charles W. Curtis and Irving Reiner, *Methods of representation theory. Vol. I*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1990.

[EH07] G. Griffith Elder and Jeffrey J. Hooper, *On wild ramification in quaternion extensions*, J. Théor. Nombres Bordeaux **19** (2007), no. 1, 101–124.

[Eld] G. Griffith Elder, *One-dimensional elementary abelian extensions of local fields*, arXiv.org:math/0511174 (2005-11-15).

[Eld95] ———, *Galois module structure of ideals in wildly ramified cyclic extensions of degree p^2*, Ann. Inst. Fourier (Grenoble) **45** (1995), no. 3, 625–647.

[Eld02] ———, *On Galois structure of the integers in cyclic extensions of local number fields*, J. Théor. Nombres Bordeaux **14** (2002), no. 1, 113–149.

[Eld06] ———, *The Galois structure of ambiguous ideals in cyclic extensions of degree 8*, Noncommutative algebra and geometry, Lect. Notes Pure Appl. Math., vol. 243, Chapman & Hall/CRC, Boca Raton, FL, 2006, pp. 63–89.

[FV02] I. B. Fesenko and S. V. Vostokov, *Local fields and their extensions*, second ed., Translations of Mathematical Monographs, vol. 121, American Mathematical Society, Providence, RI, 2002.

[Lan84] Serge Lang, *Algebra*, second ed., Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, 1984.

[Noe32] E. Noether, *Normalbasis bei Körpern ohne höhere Verzweigung*, J. Reine Angew. Math. **167** (1932), 147–152.

[RCVSM90] Martha Rzedowski-Calderón, Gabriel D. Villa Salvador, and Manohar L. Madan, *Galois module structure of rings of integers*, Math. Z. **204** (1990), no. 3, 401–424.

[Ser79] J-P. Serre, *Local fields*, Springer-Verlag, New York, 1979.

[Wym69] B. Wyman, *Wildly ramified gamma extensions*, Am. J. Math. **91** (1969), 135–152.